# The Mutilated Checkerboard in Set Theory

John McCarthy

Computer Science Department
Stanford University
jmc@cs. stanford.edu
http://www-formal.stanford.edu/jmc/

An 8 by 8 checkerboard with two diagonally opposite squares removed cannot be covered by dominoes each of which covers two rectilinearly adjacent squares. We present a set theory description of the proposition and an informal proof that the covering is impossible. While no present system that I know of will accept either the formal description or the proof, I claim that both should be admitted in any *heavy duty set theory.*[1]

We have the definitions

$$Board = Z8 \times Z8, \tag{1}$$

$$mutilated\text{-}board = Board - \{(0,0),(7,7)\}, \tag{2}$$

$$\begin{array}{l} domino\text{-}on\text{-}board(x) \equiv (x \subset Board) \wedge card(x) = 2 \\ \wedge(\forall x1\ x2)(x1 \neq x2 \wedge x1 \in x \wedge x2 \in x \\ \supset adjacent(x1,x2)) \end{array} \tag{3}$$

and

$$\begin{array}{l} adjacent(x1,x2) \equiv |c(x1,1) - c(x2,1)| = 1 \\ \wedge c(x1,2) = c(x2,2) \\ \vee |c(x1,2) - c(x2,2)| = 1 \wedge c(x1,1) = c(x2,1). \end{array} \tag{4}$$

If we are willing to be slightly tricky, we can write more compactly

[1] The Mizar theorem prover essentially accepts the definitions, but seems to be far from providing a proof.

$$adjacent(x1, x2) \equiv |c(x1, 1) - c(x2, 1)| + |c(x1, 2) - c(x2, 2)| = 1, \quad (5)$$

but then the proof might not be so obvious to the program.

Next we have

$$\begin{array}{l} partial\text{-}covering(z) \\ \equiv (\forall x)(x \in z \supset domino\text{-}on\text{-}board(x)) \\ \wedge(\forall x\ y)(x \in z \wedge y \in z \supset x = y \vee x \cap y = \{\}) \end{array} \quad (6)$$

**Theorem**:

$$\neg(\exists z)(partial\text{-}covering(z) \wedge \bigcup z = mutilated\text{-}board) \quad (7)$$

**Proof:**

We define

$$x \in Board \supset color(x) = rem(c(x, 1) + c(x, 2), 2) \quad (8)$$

$$\begin{array}{l} domino\text{-}on\text{-}board(x) \supset \\ (\exists u\ v)(u \in x \wedge v \in x \wedge color(u) = 0 \wedge color(v) = 1), \end{array} \quad (9)$$

$$\begin{array}{l} partial\text{-}covering(z) \supset \\ card(\{u \in \bigcup z | color(u) = 0\}) \\ = card(\{u \in \bigcup z | color(u) = 1\}), \end{array} \quad (10)$$

$$\begin{array}{l} card(\{u \in mutilated\text{-}board | color(u) = 0\}) \\ \neq card(\{u \in mutilated\text{-}board | color(u) = 1\}), \end{array} \quad (11)$$

and finally

$$\neg(\exists z)(partial\text{-}covering(z) \quad \wedge \quad mutilated\text{-}board = \bigcup z) \quad (12)$$

**Q.E.D.**