Development of Analysis in the QED Project

F. Javier Thayer*

1 Introduction

The QED project is a cooperative effort in the area of automated mathematical reasoning. Many goals have been suggested for this common effort. A common element in most of the suggested goals is the construction of a rigorous mathematics database. By rigorous I mean that each entry in the database is a valid theorem stated in some formal theory.

Following is a summary of my presentation on Formalized Analysis at the 2nd QED workshop held in Warsaw, Poland July 20 to 22. My focus in the discussion is the styles of mathematical development appropriate for formalizing analysis in the QED project. I believe the answer to this question is time dependent.

2 Time Scales

In discussing the QED project and its goals, I suggest we establish some time scales. My subjective values for these time horizons are short term (6 months to 2 years), medium term (2 to 5 years), long term (5 to 15) years and the distant future 15 years or more. Though QED is clearly a project for the long term or maybe even the distant future, setting achievable short and medium term goals is crucial for the project for a number of reasons. Shorter-term goals provide:

- Feedback for testing ideas.
- Motivation for developers.
- Reassurance for funders.

*e-mail: `jt@linus.mitre.org`

- Useful applications, in education, symbolic mathematics and formal methods.

3 Approaches to Analysis

The terms "Big Theory" and "Little Theory" have been coined by Farmer, Guttman and Thayer to describe certain styles of mathematical development. I will only mention here some characteristics of each of these styles. For the Big Theory style there is usually a well understood formalism in which all reasoning is imbedded. Moreover, no special logical artifices (such as theory interpretations) are needed to apply theorems about structures since structures are first-class objects. I am using structure here in the sense of a structure such as a vector space, or topological space. For the Little Theories style one considers a large number of "small" axiomatic theories interrelated by theory interpretations. Structures such as vector spaces and so on are dealt with as axiomatic theories. One considers one instance or at best a small number of instances of such structures.

4 Big Theory: Advantages and Disadvantages

I would expect any development style using the Big Theory approach to be based on some form of set theory. This is an advantage, since foundational prerequisites for the working mathematician are almost zero. Regarding the Bourbaki approach as the prototypical "big theory" style, its is clear that the Big Theory style is conceptually elegant, extremely refined, general and highly traditional.

The biggest advantage of the Big Theory approach, is that quantification over theories is straightforward. This provides a whole range of ideas and techniques not easily available in the little theories style. For instance, topological methods can be applied to families of structures, to build and study structures such as fibre bundles, sheaves, and deformations. These ideas are an essential part of the analyst's toolkit.

Though the Big Theory approach allows for enormous flexibility, and the application of a whole range of techniques (as seen above), it has various drawbacks. One minor drawback, is that "big theory" has to be developed to the point where one can begin to define structures either as tuples or by some other artifice. Another drawback, is that the mathematical universe is too homogeneous. This makes pattern matching, and consequently application of results considerably more difficult. One way of dealing with this is to superimpose a sorting structure on the universe to facilitate pattern matching.

5 Little Theories: Advantages and Disadvantages

Little Theories provide pedagogically compact, often insightful approaches to various areas in mathematics. Basically, write down the axioms and you're in business. This makes short or medium term goals attainable. Similarly, much of analysis that only involves reasoning with inequalities, fits nicely into little theories since we are not using more difficult topological properties of mappings. Finally, the Little Theories approach fits in nicely with Type Theory. This provides syntactic cues for matching and theorem application.

On the negative side, Type Theory is inflexible. For instance, completions (algebraic and completions in particular) and other extensions (such as adjunction of points at infinity to a topological space) are very hard to deal with. Another difficulty of Little Theories is the need for special artifices to compensate for lack of machinery. For instance, functorial concepts cannot be easily accommodated within the little theories approach, so that topological arguments based on homology cannot be used.

6 Discussion

The presentation was followed by a discussion that focused on a number of issues including the suitability of Type Theory for mechanized mathematics. Though my presentation was not directly focused on types, there was some concern that by favoring the Big Theory approach, I argued against types. Although I do not consider types (or sorts) to be useful in the long term, I believe there use is an extremely helpful adjunct to the Little Theories style which in the shorter term is extremely valuable.

7 Summary

The Little Theories approach is pedagogically extremely useful, and in the short and medium term is capable of providing a framework for development of automated mathematics. It can work well with existing symbolic mathematics systems providing payoff in education and possibly other areas. As a long term goal, however, I believe that QED must include in its mathematical database, general theorems on topology and geometry, theorems about families of structures, theorems about functors such as enveloping algebras and so on. This is a powerful argument that some big theory approach to QED is required.