Indefiniteness

Randall Holmes*
Math. Dept., Boise State Univ.

Abstract

The general topic of *undefined terms* was discussed and basic approaches were outlined.

1 Approaches

1.1 History

Russell's theory of descriptions, formulated in response to the views of Meinong, stands at the beginning of the topic.

1.2 Alternate Logics

Systems of free logic with an existence predicate (described in a preprint *Undefinedness* of Feferman) implement the logic of Russell's description operator, or, alternatively, can implement the approach of Meinong which Russell was trying to discredit. In either approach, the range of quantifiers is restricted to the objects which exist; in a "Russellian" approach free variables also range only over existing objects, whereas in a "Meinongian" approach free variables range over all objects, existent or otherwise.

These logics are good at handling partial functions. The PF logic of the IMPS theorem prover is a (Russellian) logic of this kind.

The common characteristic of these logics is that rules for well-formedness of terms can be kept simple, but well-formed terms do not necessarily denote, and an existence predicate is available to capture this information.

*e-mail: `holmes@math.idbsu.edu`

1.3 Typing Approaches

In this family of approaches, one adopts a type system designed to keep *undefined terms* ill-formed. This approach appears to require subtyping (for example, if we are defining division on the reals, we need the type of nonzero reals for the typing conditions). This would appear to lead to complex, even undecidable type schemes; this might not be a problem if one was already committed (as in some constructive type theories) to a very complex typing scheme.

The PF logic of IMPS is not a scheme of this kind, though it is the logic of a type theory; in PF logic, one has well-formed terms which do not denote and an existence predicate (this distinction was brought out in the discussion)

1.4 Default value approaches

In these approaches, well-formedness of terms is kept simple and all terms are treated as denoting. Terms which would normally be regarded as undefined are assigned "default values". A spectrum of approaches exists, ranging from assigning the same default value to every *undefined term* (or the same default value to every undefined term of each individual type) to an approach in which one carefully avoids providing any information about default values except that they exist. One tries to choose default values so as to keep theorems simple, but also keep them familiar-looking; it is probably a good idea to avoid having too many surprising theorems resulting from one's default convention. An example given by Boyer is a convention used by Morse in a treatment of von Neumann-Gödel-Bernays set theory (with proper classes) in which the universal class (which is a first-class object but not an element of any set) is used as the default value in all cases; this works surprisingly well in many cases.

These approaches do not interact well with the use of pointwise operations on partial functions (for example, consider the sum of the functions $x!- > 1/x$ and the constant 1 on the one hand and the function $x!- > (1/x) + 1$ on the other; suppose that *undefined values* are set to 0; these two functions have different values at 0 and so are distinct). This suggests the use of a special error value for undefined values of functions and restricting one's attention to "strict" functions (those which send the error value to the error value); such an approach would culminate in something like the semantics of Scott's models of the lambda-calculus. Notice that Morse's default value approach to set theory described above also handles this correctly.

2 Relation to QED

Different approaches to undefined terms create an obstruction to exchange of information between theorem provers which has nothing to do with underlying mathematical issues (there are likely to be other such obstructions arising from differences of mathematical convention rather than content).

The character of different provers may dictate which approach is most convenient: for example, in the prover which Holmes is working on, an equational prover without a built-in type system or an easy way to express side conditions, a default value approach was most natural. Thus, a commitment to a single approach throughout QED seems undesirable.

Holmes suggested the design of a menu of standard approaches to undefined terms and a set of protocols for translation between these approaches. An example: translation from a type-based approach to an existence predicate based approach should be easy.