What Can QED Offer to Mathematics?

QED Workshop II, Warsaw, July 20–22, 1995

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1 Correctness

The main argument why QED can be of great practical use is the ongoing striving for correctness in mathematics at a level as high as possible: a system like QED is the answer of our time to that striving. That this correctness problem is not sufficiently solved in traditional mathematics can be seen in the history of false theorems and false proofs.

False theorems: The perhaps best studied example of a theorem that had to be corrected again and again is Euler's polyhedron theorem. Imre Lakatos[1] has described the cycle of stating the theorem, proving it, and finding counterexamples, then going into the next loop with a refined version of the theorem. How can it be possible that there are counterexamples to a proved theorem? This can be due to inaccurate definitions, hidden assumptions, or invalid arguments. The main problem in the case of Euler's polyhedron theorem has been what to consider as a polyhedron. A system like QED, however, would not accept any inaccurate definitions, hidden assumptions, or invalid arguments, but it forces a human user to be precise.

False proofs: While the problem of false theorems is much more grave than that of false proofs, false proofs, which can be patched, occur much more often. This problem arises since mathematical proofs are normally not given on a formal logical level, but on an intuitive level where the ideas how to build a proof are conveyed rather than the proofs itself. On such a level it is possible that a proof is not understandable or that certain non-trivial gaps

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[1]see: Imre Lakatos. *Proofs and Refutations*. Cambridge University Press, 1976.

may occur (e.g. in the proof of Fermat's Last Theorem), for which it is not clear whether they can be closed or not.

QED can support the author, the reviewer, and the user of a theorem against false theorems as well as against false proofs. For each theorem must exist a machine-checked proof.

2 Proof Presentation

Proving mathematical theorems is a social process. The verification of mathematical arguments is a key activity in mathematics and hence very sensitive for the development of mathematics. Therefore it is not a good idea to try to replace this process by checking proofs in QED. On the contrary, QED should be considered as a supplement to the traditional process of proof checking and supporting this process. In particular QED should support different levels of the proof display (such as the logic level, assertional level, expert level with a user model, interactive proof display, or even multi-media facilities).

3 Information Retrieval

While correctness and proof presentation may not be the main gains mathematicians can draw from QED, they may be very interested in using a large data base of definitions and theorems, when it is well-organized and easy to use. A large data base of mathematical facts in QED can easily provide useful information like the interdependence of certain axioms, definitions and theorem, which is hard to obtain without computer support.

4 Future Prospects

In order to come up with a system that finds a broad distribution, QED eventually has to enable a human user to develop a machine-checked proof at least as easily as to do that on paper. Therefore it is necessary to provide strong support in the proof search in more areas. In order to do that standard automated theorem proving techniques as well as more high-level oriented approaches like proof planning and analogy should be used.

But even when this goal is reached a lot of additional functionality can be integrated in QED, for instance support for some standard procedure of mathematics, like the approach of changing the definitions in order to get the right theorem.

5 Discussion

In the discussion the following points were made:

- Correctness may be too weak an argument for mathematicians when confronted with the QED-approach, since at first a lot of basic work has to be done before they can really start. Furthermore errors in papers are normally not too important. Therefore QED has to do a lot of preliminary work. In Mizar, for instance, thirty master theses were written. This led not only to a profound amount of formal mathematical knowledge, but had also the interesting side effect that the relationship between a teacher and a student changed a lot: the teacher does not force the student to be precise, but the system does and the teacher has the role of assisting the student. In order to increase the acceptance of QED, it would be ideal to expose students to formal proving already in an early stage of their academic training.

- It would be fine to have a collection of well-documented examples for false theorems and false proofs.

- A main practical argument for mathematicians to use QED could be the speed of publications. If the article is formally written in QED, cumbersome proof reading can be done mechanically and the referees can concentrate on the relevance of the paper rather than on the correctness.